On large deviation rates for sums associated with Galton-Watson processes [1]

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Abstract

Given a super-critical Galton-Watson process $\{Z_n\}$ and a positive sequence $\{\epsilon_n\}$, we study the limiting behaviors of $P(S_{Z_n}/Z_n \geq \epsilon_n)$ with sums S_n of i.i.d. random variables X_i and $m = E[Z_1]$. We assume that we are in Schröder case with $EZ_1 \log Z_1 < \infty$ and X_1 is in the domain of attraction of an α-stable law with $0 < \alpha < 2$. As a by-product, when Z_1 is sub-exponentially distributed, we further obtain the convergence rate of $\frac{Z_{n+1}}{Z_n}$ to m as $n \to \infty$.

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Abbreviated Title: LDP for sums

1 Introduction and Main Results

1.1 Motivation

Let $Z = (Z_n)_{n\geq 1}$ be a super-critical Galton-Watson process with $Z_0 = 1$ and offspring distribution $\{p_k : k \geq 0\}$. Define $m = \sum_{k\geq 1} kp_k > 1$. We assume in this paper that $p_0 = 0$ and $0 < p_1 < 1$.

It is known that $Z_{n+1}/Z_n \overset{a.s.}{\to} m$ and Z_{n+1}/Z_n is the so-called Lotka-Nagaev estimator of m; see Nagaev [14]. This estimator has been used in studying amplification rate and the initial number of molecules for amplification process in a quantitative polymerase chain reaction experiment; see [12, 13] and [18]. Concerning the Bahadur efficiency of the estimator leads to investigating the large deviation behaviors of Z_{n+1}/Z_n. In fact, it was proved in [14] that if $\sigma^2 = Var(Z_1) \in (0, \infty)$, then

$$\lim_{n\to\infty} P\left(m^{n/2}\left(\frac{Z_{n+1}}{Z_n} - m\right) < x\right) = \int_0^\infty \Phi\left(\frac{x\sqrt{u}}{\sigma}\right)\omega(u)du, \tag{1}$$

where Φ is the standard normal distribution function and ω denotes the continuous density function of $W :\overset{a.s.}{=} \lim_{n\to\infty} Z_n/m^n$. In [1], Athreya showed that if $p_1 m^r > 1$ and $E[Z_1^{2r+\delta}] < \infty$ for some $r \geq 1$ and $\delta > 0$, then

$$\lim_{n\to\infty} \frac{1}{p_1^n} P\left(\left|\frac{Z_{n+1}}{Z_n} - m\right| \geq \epsilon\right) \text{ exits finitely;}$$

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see also [3]. Later, Ney and Vidyashankar [16] weakened the assumption and were able to obtain the rate of convergence of Lotka-Nagaev estimator by studying the asymptotic properties of harmonic moments of Z_n, where it was assumed that $P(Z_1 \geq x) \sim ax^{1-\eta}$ for some $\eta > 2$ and $a > 0$. See [17] for some further results.

Recently, Fleischmann and Wachtel [11] considered a generalization of above problem by studying sums indexed by Z; see also [17]. More precisely, let $X = (X_n)_{n\geq 1}$ denote a family of i.i.d. real-valued random variables. They investigated the large deviation probabilities for S_{Z_n}/Z_n: the convergence rate of

$$P\left(\frac{S_{Z_n}}{Z_n} \geq \epsilon_n\right),$$

as $n \to \infty$, where $\epsilon_n \to 0$ is a positive sequence and

$$S_n := X_1 + X_2 + \cdots + X_n.$$

In fact, if $X_1 \stackrel{d}{=} Z_1 - m$, then

$$\frac{S_{Z_n}}{Z_n} \stackrel{d}{=} \frac{Z_{n+1}}{Z_n} - m.$$

The assumption in [11] is that $E[Z_1 \log Z_1] < \infty$, $E[X_1^2] < \infty$ and $P(X_1 \geq x) \sim ax^{-\eta}$ for some $\eta > 2$, which implies that X_1 is in the domain of attraction of normal distributions.

Motivated by above mentioned works, the main purpose of this paper is trying to study the convergence rates of Z_{n+1}/Z_n under weaker conditions. We shall use the framework of [11] but we assume that $E[Z_1 \log Z_1] < \infty$ and X_1 is in the domain of attraction of a stable law; see Assumptions A and B below. Then we answer a question in [11]; see (a) in Remark 11 there. In particular, we further obtain the convergence rate of Z_{n+1}/Z_n under the assumption $P(Z_1 > x) \sim L(x)x^{-\beta}$ for some $1 < \beta < 2$ and some slowly varying function L, which partially improves Theorem 3 in [16].

For proofs, we shall use the strategy of [11]. However, our arguments are deeply involved because of the lack of high moments and the perturbations of slowly varying functions. We overcome those difficulties by using Fuk-Nagaev's inequalities, estimation of growth of random walks, large deviation probabilities for sums under sub-exponentiallity and establishing the asymptotic properties of

$$E[Z_n^{-t} L(\epsilon_n Z_n)], \quad t > 0, \quad \text{as } n \to \infty. \tag{2}$$

In the next section, Section 1.2, we will give our basic assumptions on Z and X. Our main results will be presented in Section 1.3. We prove Fuk-Nagaev's inequalities and establish the asymptotic properties of (2) in Section 2. The proofs of main results will be given in Section 3. With C, c, etc., we denote positive constants which might change from line to line.

1.2 Basic Assumptions

Define $F(x) = P(X_1 \leq x)$. We make the following assumption:

Assumption A:

- $P(X_1 \geq x) \sim x^{-\beta} L(x)$, where $\beta > 0$ and L is a slowly varying function;
- If $\epsilon_n \to 0$, we assume that L is bounded away from 0 and ∞ on every compact subset of $[0, \infty)$.
- X_1 is in the domain of attraction of an α-stable law with $0 < \alpha < 2$;

- $E[X_1] = 0$ if $1 < \alpha < 2$;
- $E[Z_1 \log Z_1] < \infty$;
- $p_0 = 0, 0 < p_1 < 1$.

From the Assumption, it is easy to see that $\alpha \le \beta$. The last term in the Assumption means that we are in the Schröder case. In fact, we only need to assume $0 < p_0 + p_1 < 1$.

Remark 1.1. *The second term in the Assumption is technical. In fact, by Theorem 1.5.6 in [4] for any $\eta > 0$ and $a > 0$, there exist two positive constants C_η such that, for any $y > a$, $z > a$,*

$$\frac{L(z)}{L(y)} \le C_\eta \max\left(\left(\frac{z}{y}\right)^{\eta}, \left(\frac{z}{y}\right)^{-\eta}\right). \tag{3}$$

And if L is bounded away from 0 and ∞ on every compact subset of $[0, \infty)$, then (3) holds for any $y > 0, z > 0$.

Remark 1.2. *Under Assumption A we have that there exists a function $b(k)$ of regular variation of index $1/\alpha$ such that*

$$b(k)^{-1} S_k \xrightarrow{d} U_s, \tag{4}$$

where U_s is an α-stable random variable; see [9] and [21]. Without loss of generality, we may and will assume that function b is continuous and monotonically increasing from $\mathbb{R}^+$ onto $\mathbb{R}^+$ and $b(0) = 0$; see [9]. We also have that

$$b(x) = x^{1/\alpha} s(x), \quad x > 0,$$

where $s : (0, \infty) \to (0, \infty)$ is a slowly varying function. Then (3) also holds for s with $y \ge 1, z \ge 1$.

Define

$$\mu(1; x) = \int_{-x}^{x} y F(dy), \quad \mu(2; x) = \int_{-x}^{x} y^2 F(dy). \tag{5}$$

Under Assumption A, by arguments in [9], we have as $x \to +\infty$,

$$\frac{1 - F(x)}{1 - F(x) + F(-x)} \to p_+, \quad \frac{F(-x)}{1 - F(x) + F(-x)} \to p_-, \quad p_+ + p_- = 1 \tag{6}$$

and

$$\frac{x^2[1 - F(x) + F(-x)]}{\mu(2; x)} \to \frac{2 - \alpha}{\alpha}, \qquad \mu(2; x) \sim \begin{cases} \frac{\alpha}{2-\alpha} x^{2-\alpha} R(x), & \text{if } p_+ = 0; \\ \frac{\beta p_+}{2-\beta} x^{2-\beta} L(x), & \text{if } 0 < p_+ < 1; \\ \frac{\beta}{2-\beta} x^{2-\beta} L(x), & \text{if } p_+ = 1, \end{cases} \tag{7}$$

where R is a slowly varying function. Furthermore, the function b in (4) must satisfy: as $x \to +\infty$,

$$x[1 - F(b(x))] \to C p_+ \frac{2 - \alpha}{\alpha}, \quad x F(-b(x)) \to C p_- \frac{2 - \alpha}{\alpha}; \tag{8}$$

see (5.25) in [9]. In particular, it is implied in above that if $p_+ = 0$, then $F(-x) \sim x^{-\alpha} R(x)$ as $x \to +\infty$. Then for some technically reasons, we also need to make the following assumptions.

Assumption B:

- U_s is strictly stable;
- If $1 < \alpha < 2$, we assume that $\liminf_{x\to+\infty} s(x) \in (0, +\infty]$;
- If $0 < p_+ < 1$ and $\alpha = 1$, we assume that $\mu(1; x) = 0$ for all $x > 0$;
- If $p_+ = 0$, we assume $\alpha < \beta$;
- If $1 < \alpha < 2$ and $p_+ > 0$, we assume

$$\limsup_{n\to+\infty} \frac{F(-b(n)/[\log n]^{1/\alpha})}{(\log n)F(-b(n))} \le 1.$$

Remark 1.3. *The assumption that U_s is strictly stable implies that, when $\alpha = 1$, we must have $\alpha = \beta$ and the skewness parameter of U_s is 0. The 2nd term in Assumption B will be used to deduce (46) which is required in Lemma 3.3. The 3rd term is used in Step 2 in Lemma 3.4 to find a good upper bound for $P(x)$, which appears in Theorem 1.2 in [15]. The last two terms are required in Theorems 9.2 and 9.3 in [7], which are needed in our proofs.*

From now on, Assumptions A and B are in force.

1.3 Main Results

Before presenting the main results, we first introduce some notation. Recall $b(x)$ from (4). Define $J(x) = xb(x)^{-1}$ and

$$l(x) = \inf\{y \in [0, \infty) : J(y) > x\}.$$

According to Theorem 1.5.12 in [4], $l(x)$ is an asymptotic inverse of J; i.e.;

$$l(J(x)) \sim J(l(x)) \sim x, \text{ as } x \to +\infty.$$

Define $l(\epsilon_n^{-1}) = l_n$. Note that l is also regular varying function with index $\frac{\alpha-1}{\alpha}$. Denote by $f(s)$ the generating function of our offspring law. Define γ (Schröder constant) by

$$f'(0) = m^{-\gamma} = p_1.$$

For $1 < \alpha < 2$ and $\alpha < \beta$, let

$$\chi_n := \frac{l_n^{\gamma-\beta} m^{(\beta-1-\gamma)n} b(l_n)^\beta}{L(l_n^{-1} b(l_n) m^n)} = \frac{b(l_n)^\gamma}{(\epsilon_n m^n)^{\gamma-\beta} L(\epsilon_n m^n) m^n}.$$

For $0 \le t < \gamma + 1$, define

$$I_t = \int_0^\infty u^{1-t} \omega(u) du. \tag{9}$$

Remark 1.4. *As $u \to 0+$, there exist constants $0 < C_1 < C_2 < \infty$ such that*

$$C_1 < \frac{\omega(u)}{u^{\gamma-1}} < C_2. \tag{10}$$

See [8], [5] and references therein for related results. So the assumption $E[Z_1 \log Z_1] < \infty$, together with (10), implies that I_t is finite; see Theorem 8.12.7 in [4].

We are ready to present our main results. As illustrated in [16], there is a "phase transition" in rates depending on γ. Thus we will have three different cases in regard to γ and β. We first consider the case of $\gamma > \beta - 1$.

Theorem 1.5. *Let $0 < \alpha < 1$. Assume that $\epsilon_n m^n b(m^n)^{-1} \to +\infty$ and $\epsilon_n \to +\infty$ as $n \to \infty$. If $\gamma > \beta - 1$, then*

$$\lim_{n\to\infty} m^{(\beta-1)n}\epsilon_n^\beta L(\epsilon_n m^n)^{-1} P(S_{Z_n}/Z_n \geq \epsilon_n) = I_\beta. \tag{11}$$

Theorem 1.6. *Let $1 \leq \alpha < 2$. Assume that $\epsilon_n m^n b(m^n)^{-1} \to +\infty$ as $n \to \infty$ and $\gamma > \beta - 1$.*

(i) Assume $1 < \alpha < 2$, $p_+ = 0$ and $\epsilon_n \to 0$. If $\lim_{n\to\infty} \chi_n = 0$, then (11) holds.

(ii) Assume $1 < \alpha < 2$, $p_+ = 0$ and $\epsilon_n \to 0$. If $\lim_{n\to\infty} \chi_n = \infty$, then

$$\begin{aligned} V_I &\leq \varliminf_{n\to\infty} l_n^{-\gamma} m^{\gamma n} P(S_{Z_n}/Z_n \geq \epsilon_n) \\ &\leq \varlimsup_{n\to\infty} l_n^{-\gamma} m^{\gamma n} P(S_{Z_n}/Z_n \geq \epsilon_n) \leq V_S, \end{aligned} \tag{12}$$

where

$$\begin{aligned} V_I &= \varliminf_{u\downarrow 0} u^{1-\gamma}\omega(u)\int_0^\infty u^{\gamma-1} P(U_s \geq u^{\frac{\alpha-1}{\alpha}})du, \\ V_S &= \varlimsup_{u\downarrow 0} u^{1-\gamma}\omega(u)\int_0^\infty u^{\gamma-1} P(U_s \geq u^{\frac{\alpha-1}{\alpha}})du. \end{aligned}$$

(iii) Assume $1 < \alpha < 2$, $p_+ = 0$ and $\epsilon_n \to 0$. If $\lim_{n\to\infty} \chi_n = y \in (0, \infty)$, then

$$\begin{aligned} V_I + yI_\beta &\leq \varliminf_{n\to\infty} l_n^{-\gamma} m^{\gamma n} P(S_{Z_n}/Z_n \geq \epsilon_n) \\ &\leq \varlimsup_{n\to\infty} l_n^{-\gamma} m^{\gamma n} P(S_{Z_n}/Z_n \geq \epsilon_n) \leq V_S + yI_\beta, \end{aligned}$$

(iv) Assume $p_+ > 0$ and $\epsilon_n \to \epsilon \in (0, \infty)$. Then (11) holds.

Remark 1.7. *The assumption $p_+ = 0$ implies that U_s is a spectrally negative α-stable random variable with mean 0 and skewness parameter -1. By (1.2.11) in [21], we have*

$$\int_0^\infty u^{\gamma-1} P(U_s \geq u^{\frac{\alpha-1}{\alpha}})du < \infty.$$

As an application of (iv) in above theorem by taking $\epsilon_n = \epsilon$, we immediately get the following result, which improves the corresponding result in Theorem 3 in [16], where it is assumed that L is a constant function.

Corollary 1.8. *If $P(Z_1 > x) \sim x^{-\beta}L(x)$ for $1 < \beta < 2$ and $\gamma > \beta - 1$, then*

$$\lim_{n\to\infty} m^{(\beta-1)n} L(m^n)^{-1} P\left(\frac{Z_{n+1}}{Z_n} - m \geq \epsilon\right) = I_\beta \epsilon^{-\beta}. \tag{13}$$

Remark 1.9. *In fact, by (52) below, one may prove that*

$$\lim_{n\to\infty} m^{(\beta-1)n} L(m^n)^{-1} P\left(m - \frac{Z_{n+1}}{Z_n} \geq \epsilon\right) = 0.$$

Proof. (iv) in Theorem 1.6 implies (13). □

Next, we consider the case of $\gamma = \beta - 1$. Let d be the greatest common divisor of the set $\{j - i : i \neq j, p_j p_i > 0\}$.

Theorem 1.10. *Suppose $0<\alpha<1$ and $\beta>1$. Assume that $\epsilon_n m^n b(m^n)^{-1}\to+\infty$ and $\epsilon_n\to+\infty$ as $n\to\infty$. If $\gamma=\beta-1$, then*

$$\begin{aligned} d\liminf_{u\downarrow 0} u^{1-\gamma}\omega(u) &\le \liminf_{n\to\infty}\frac{\epsilon_n^\beta P(S_{Z_n}/Z_n\ge\epsilon_n)}{\sum_{1\le k\le m^n}\frac{L(\epsilon_n k)}{km^{\gamma n}}} \\ &\le \limsup_{n\to\infty}\frac{\epsilon_n^\beta P(S_{Z_n}/Z_n\ge\epsilon_n)}{\sum_{1\le k\le m^n}\frac{L(\epsilon_n k)}{km^{\gamma n}}} \le d\limsup_{u\downarrow 0} u^{1-\gamma}\omega(u). \end{aligned} \tag{14}$$

Define

$$\pi_n=\frac{l_n^\gamma\epsilon_n^\beta}{\sum_{1\le k\le m^n}\frac{L(\epsilon_n k)}{k}}.$$

Theorem 1.11. *Let $1<\alpha<2$. Assume that $\epsilon_n\to 0$, $\epsilon_n m^n b(m^n)^{-1}\to+\infty$.*

(i) Assume $p_+=0$ and $\gamma=\beta-1$. If $\pi_n\to 0$, then (14) holds.

(ii) Assume $p_+=0$ and $\gamma=\beta-1$. If $\pi_n\to+\infty$, then (12) holds.

(iii) Assume $p_+=0$ and $\gamma=\beta-1$. If $\pi_n\to y\in(0,\infty)$, then

$$\begin{aligned} V_I+yd\liminf_{u\downarrow 0}u^{1-\gamma}\omega(u) &\le \liminf_{n\to\infty} l_n^{-\gamma}m^{\gamma n}P(S_{Z_n}/Z_n\ge\epsilon_n) \\ &\le \limsup_{n\to\infty} l_n^{-\gamma}m^{\gamma n}P(S_{Z_n}/Z_n\ge\epsilon_n)\le V_S+yd\limsup_{u\downarrow 0}u^{1-\gamma}\omega(u), \end{aligned}$$

(iv) Assume $p_+>0$ and $\gamma=\beta-1$. Then (14) holds with ϵ_n replaced by any $\epsilon>0$.

Remark 1.12. *If L is a constant function, then (14) can be replaced by*

$$\lim_{n\to\infty} n^{-1}\epsilon_n^\beta m^{\gamma n}P(S_{Z_n}/Z_n\ge\epsilon_n)=\frac{1}{\Gamma(\beta-1)}\int_1^m Q(E[e^{-vW}])v^{\beta-2}dv,$$

where

$$Q(s)=\sum_{k=1}q_k s^k=\lim_{n\to\infty}\frac{f_n(s)}{m^{-\gamma n}},\quad 0\le s<1,\quad q_k=\lim_{n\to\infty}P(Z_n=k)m^{\gamma n} \tag{15}$$

and f_n denotes the iterates of f. See Proposition 2 in [1] for $Q(s)$ and $(q_k)_{k\ge1}$. The key is the limiting behavior of $E[Z_n^{-\gamma}L(\epsilon_n Z_n)]$ as $n\to\infty$; see Theorem 1 in [16] and Remark 2.3 below in this paper.

Finally, we consider the case of $\gamma<\beta-1$.

Theorem 1.13. *If $1<\alpha<2$ and $\gamma<\beta-1$ or $E[X_1^{1+\gamma}1_{\{X_1>0\}}]<\infty$, then for any $\epsilon>0$,*

$$\lim_{n\to\infty}m^{\gamma n}P(S_{Z_n}/Z_n\ge\epsilon)=\sum_{k\ge1}q_kP(S_k\ge\epsilon k).$$

Corollary 1.14. *If $P(Z_1>x)\sim x^{-\beta}L(x)$ for $1<\beta<2$ and $\gamma<\beta-1$ or $E[Z_1^{1+\gamma}]<\infty$, then*

$$\lim_{n\to\infty}m^{\gamma n}P\left(\left|\frac{Z_{n+1}}{Z_n}-m\right|\ge\epsilon\right)=\sum_{k\ge1}q_k\phi(k,\epsilon),$$

where $\phi(k,\epsilon)=P(|\frac{1}{k}\sum_{i=1}^k\xi_i-m|>\epsilon)$ and $(\xi_i)_{i\ge1}$ are i.i.d. random variables with the same distribution as Z_1.

Remark 1.15. *When L is a constant function and $P(Z_1 > x) \sim x^{-\beta}L$, the above result has been proved in [16]. Theorem 1 and Corollary 1 in [1] also proved the same result under the assumption $E[Z_1^{2a+\delta}] < \infty$ and $p_1 m^a > 1$ for some $a \geq 1$ and $\delta > 0$.*

We also generalize (1) to the stable setting.

Theorem 1.16. *Assume that $0 < \alpha < 2$. If $\epsilon_n m^n b(m^n)^{-1} \to x \in (-\infty, +\infty)$, then*

$$\lim_{n\to\infty} P(S_{Z_n}/Z_n \geq \epsilon_n) = \int_0^\infty P\left(U_s \geq u^{\frac{\alpha-1}{\alpha}} x\right) \omega(u) du. \tag{16}$$

As an application of above theorem, the following result generalizes (1); see Theorem 3 in [14].

Corollary 1.17. *Assume that $1 < \beta < 2$ and $P(Z_1 > x) \sim x^{-\beta}L(x)$ as $x \to +\infty$. Then for every $x \in (-\infty, +\infty)$,*

$$\lim_{n\to\infty} P\left(\frac{m^n}{b(m^n)}\left(\frac{Z_{n+1}}{Z_n} - m\right) \leq x\right) = \int_0^\infty P\left(U_s \leq u^{\frac{\beta-1}{\beta}} x\right) \omega(u) du. \tag{17}$$

Proof. Obviously, $Z_1 - m$ is in the domain of attraction of β-stable law. Using Theorem 1.16 with $\epsilon_n = xb(m^n)m^{-n}$ gives (17). $\square$

Remark 1.18. *It is possible to generalize some results above to the setting that $(X_i)_{i\geq 1}$ are not independent; see [22, 23] and references therein for related results.*

2 Preliminaries

2.1 Fuk-Nagaev inequalities

The following result is parallel to Lemma 14 in [11] where X_1 has finite variance.

Lemma 2.1. *For any $0 < \alpha < 1$, $r > 0$ and $k \geq 1$,*

$$P(S_k \geq \epsilon_n k) \leq \begin{cases} kP(X_1 \geq r^{-1}\epsilon_n k) + c_r \epsilon_n^{-\beta r} k^{(1-\beta)r}, & \beta < 1 \\ kP(X_1 \geq r^{-1}\epsilon_n k) + c_r \epsilon_n^{-tr} k^{(1-t)r}, & \beta \geq 1 \end{cases}. \tag{18}$$

hold for $t \in (\alpha, 1] \cap (\alpha, \beta)$.

Proof. By Theorem 1.1 in [15], we have for any $0 < t \leq 1$,

$$P(S_k \geq x) \leq kP(X_1 \geq y) + \exp\left\{\frac{x}{y} - \frac{x}{y}\ln\left(\frac{xy^{t-1}}{kA(t;0,y)} + 1\right)\right\}$$

with $A(t;0,y) = E[X_1^t \cdot 1_{\{0\leq X_1 \leq y\}}]$, which gives

$$P(S_k \geq \epsilon_n k) \leq kP(X_1 \geq r^{-1}\epsilon_n k) + \left(\frac{eE[X_1^t; 1_{\{0\leq X_1\leq r^{-1}\epsilon_n k\}}]}{r^{1-t}\epsilon_n^t k^{t-1}}\right)^r. \tag{19}$$

Noting that as $x \to +\infty$, $P(X_1 \geq x) \sim x^{-\beta}L(x)$, we have for $x > 1$,

$$E[X_1^t; 1_{\{0\leq X_1\leq x\}}] \leq \begin{cases} Cx^{t-\beta}, & \beta < t; \\ C_t, & t < \beta. \end{cases} \tag{20}$$

And if $x \leq 1$, obviously we have

$$E[X_1^t; 1_{\{0\leq X_1\leq x\}}] \leq C(1 \vee x^{t-\beta}). \tag{21}$$

Then if $\beta < 1$, applying (19) with $\beta < t$, together with (20) and (21), yields (18). If $\beta \geq 1$, with the help of (20) and (21), taking any $\alpha < t \leq 1$ and $r > 0$ also implies (18). $\square$

2.2 Harmonic moments

It is well-known that

$$W_n := m^{-n} Z_n \xrightarrow{a.s.} W;$$

see [10]. We further have the global limit theorem:

$$\lim_{n\to\infty} P(Z_n \geq xm^n) = \int_x^\infty \omega(t)dt, \quad x > 0. \tag{22}$$

In particular, one can deduce that for $0 < \delta < 1 < A < \infty$

$$E[(W_n)^t 1_{\{W_n<\delta\}}] \to \int_0^\delta u^t \omega(u)du, \quad t > -\gamma; \tag{23}$$

$$E[(W_n)^t 1_{\{W_n>A\}}] \to \int_A^\infty u^t \omega(u)du, \quad -\infty < t \leq 1. \tag{24}$$

We also recall here a result from Lemma 13 in [11]. There exists a constant $C > 0$ *such that*

$$P(Z_n = k) \leq C\left(\frac{1}{k} \wedge \frac{k^{\gamma-1}}{m^{\gamma n}}\right), \quad k,\, n \geq 1. \tag{25}$$

Lemma 2.2. *Assume* $\epsilon_n m^n \to \infty$*. Then as* $n \to \infty$*,*

$$E[Z_n^t L(\epsilon_n Z_n)] \sim m^{nt} L(\epsilon_n m^n) \int_0^\infty u^t \omega(u)du, \quad -\gamma < t < 1; \tag{26}$$

and

$$\begin{aligned} d \varliminf_{u\downarrow 0} u^{1-\gamma}\omega(u) &\leq \varliminf_{n\to\infty} \frac{E[Z_n^{-\gamma} L(\epsilon_n Z_n)]}{\sum_{1\leq k\leq m^n} \frac{L(\epsilon_n k)}{k m^{\gamma n}}} \\ &\leq \varlimsup_{n\to\infty} \frac{E[Z_n^{-\gamma} L(\epsilon_n Z_n)]}{\sum_{1\leq k\leq m^n} \frac{L(\epsilon_n k)}{k m^{\gamma n}}} \leq d \varlimsup_{u\downarrow 0} u^{1-\gamma}\omega(u). \end{aligned} \tag{27}$$

Proof. We first prove (26). Recall $W_n = Z_n/m^n$. Note that

$$E\left[Z_n^t L(\epsilon_n Z_n)\right] = m^{nt} L(\epsilon_n m^n) E\left[(W_n)^t \frac{L(\epsilon_n m^n W_n)}{L(\epsilon_n m^n)}\right]. \tag{28}$$

Then for $0 < \delta < 1 < A$, by (3) and (23), we have for some $0 < \eta < \gamma$ small enough,

$$E\left[(W_n)^t \frac{L(\epsilon_n m^n W_n)}{L(\epsilon_n m^n)} 1_{\{W_n<\delta\}}\right] \leq CE[(W_n)^{t-\eta} 1_{\{W_n<\delta\}}] = (1+o(1))C \int_0^\delta u^{t-\eta}\omega(u)du. \tag{29}$$

Meanwhile by Dominated Convergence Theorem, we have

$$E\left[(W_n)^t \frac{L(\epsilon_n m^n W_n)}{L(\epsilon_n m^n)} \cdot 1_{\{\delta\leq W_n\leq A\}}\right] \to \int_\delta^A u^t \omega(u)du. \tag{30}$$

Finally, using (3) with $\eta = 1 - t$, we have

$$E\left[(W_n)^t \frac{L(\epsilon_n m^n W_n)}{L(\epsilon_n m^n)} 1_{\{W_n>A\}}\right] \leq CE[W_n 1_{\{W_n>A\}}] = (1+o(1))C \int_A^\infty u\omega(u)du. \tag{31}$$

Letting $\delta \to 0$ and $A \to \infty$, together with (28), we obtain (26).

The sequel of this proof is devoted to (27). Let $\{k_n\}$ be a sequence such that $k_n \to \infty$ and $k_n = o(m^n)$. Then for any $0 < \delta \le 1$,

$$E[Z_n^{-\gamma} L(\epsilon_n Z_n)] = \left(\sum_{k<k_n} + \sum_{k_n \le k \le \delta m^n} + \sum_{k>\delta m^n} \right) \frac{L(\epsilon_n k)}{k^\gamma} P(Z_n = k) =: I_0 + I_1 + I_2.$$

By Corollary 5 in [10], we have

$$I_1 = (1+o(1))d \sum_{k_n \le k \le \delta m^n} \frac{L(\epsilon_n k)}{k^\gamma} m^{-n} \omega\left(\frac{k}{m^n}\right)$$

which is larger than

$$(1+o(1))d \inf_{u\le\delta} u^{1-\gamma}\omega(u) \sum_{k_n \le k \le \delta m^n} \frac{L(\epsilon_n k)}{k m^{\gamma n}}$$

and less than

$$(1+o(1))d \sup_{u\le\delta} u^{1-\gamma}\omega(u) \sum_{k_n \le k \le \delta m^n} \frac{L(\epsilon_n k)}{k m^{\gamma n}}.$$

On the other hand, Dominated Convergence Theorem, together with (3), tells us

$$I_2 \sim m^{-\gamma n} L(\epsilon_n m^n) \int_\delta^\infty u^{-\gamma}\omega(u)du.$$

And we have

$$\frac{Z_n^{-\gamma} L(\epsilon_n Z_n)}{m^{-\gamma n} L(\epsilon_n m^n)} 1_{\{Z_n \le \delta m^n\}} \overset{a.s.}{\to} W^{-\gamma} 1_{\{W \le \delta\}}$$

whose expectation is infinite by (10). Then Fatou's lemma yields

$$\limsup_{n\to\infty} I_2/(I_0 + I_1) = 0. \tag{32}$$

By (25), we also have

$$I_0 \le \sum_{k<k_n} \frac{L(\epsilon_n k)}{k m^{\gamma n}}. \tag{33}$$

Then one may choose k_n such that

$$\frac{\sum_{k<k_n} \frac{L(\epsilon_n k)}{k}}{\sum_{k<m^n} \frac{L(\epsilon_n k)}{k}} \to 0. \tag{34}$$

Meanwhile, one can also deduce that

$$\begin{aligned}(1+o(1))d \inf_{u\le\delta} u^{1-\gamma}\omega(u) \sum_{\delta m^n \le k \le m^n} \frac{L(\epsilon_n k)}{k m^{\gamma n}} &\le E[Z_n^{-\gamma} L(\epsilon_n Z_n) 1_{\{\delta m^n \le Z_n \le m^n\}}] \\ &\sim m^{-\gamma n} L(\epsilon_n m^n) \int_\delta^1 u^{-\gamma}\omega(u)du,\end{aligned}$$

which, together with (33), (34) and (32), gives $\limsup_{n\to\infty} I_0/I_1 = \limsup_{n\to\infty} I_2/I_1 = 0$. Thus

$$d \inf_{u<\delta} u^{1-\gamma}\omega(u) \le \underline{\lim}_{n\to\infty} \frac{E[Z_n^{-\gamma} L(\epsilon_n Z_n)]}{\sum_{1\le k \le m^n} \frac{L(\epsilon_n k)}{k m^{\gamma n}}}$$

$$\leq \overline{\lim_{n\to\infty}} \frac{E[Z_n^{-\gamma} L(\epsilon_n Z_n)]}{\sum_{1\leq k\leq m^n} \frac{L(\epsilon_n k)}{k m^{\gamma n}}} \leq d \sup_{u<\delta} u^{1-\gamma}\omega(u)$$

holds for any $\delta > 0$. Letting $\delta \to 0$ implies (27). We have completed the proof. $\square$

Remark 2.3. *Lemma 2.2 could be compared with Theorem 1 in [16] where $L = 1$. Under the assumption $E[Z_1 \ln Z_1] < \infty$, when $-\gamma < t < 0$, our result completes the one in [16]. However, when $t = -\gamma$, a precise limit is obtained in [16].*

3 Proofs

We only prove Theorems 1.5, 1.6, 1.13 and 1.16. The ideas to prove Theorems 1.10 and 1.11 are similar to Theorems 1.5 and 1.6, respectively. We omit details here.

3.1 Proof of Theorem 1.5

Lemma 3.1. *Assume that $0 < \alpha < 1$. If $\gamma > \beta - 1$, $\epsilon_n m^n b(m^n)^{-1} \to +\infty$ and $\epsilon_n \to +\infty$, then there exits $\eta > 0$ small enough such that for any $0 < \delta < 1 < A$,*

$$\limsup_{n\to\infty} \frac{\epsilon_n^{\beta}(m^n)^{(\beta-1)}}{L(\epsilon_n m^n)} \sum_{k\leq\delta m^n} P(Z_n = k)P(S_k \geq k\epsilon_n) \leq C\delta^{\gamma-\beta+1-\eta}; \tag{35}$$

$$\limsup_{n\to\infty} \frac{\epsilon_n^{\beta}(m^n)^{(\beta-1)}}{L(\epsilon_n m^n)} \sum_{k\geq A m^n} P(Z_n = k)P(S_k \geq k\epsilon_n) \leq C\int_A^\infty u\omega(u)du. \tag{36}$$

Proof. We first prove (35). Consider the case of $\beta < 1$. Applying (3) with $0 < \eta < \gamma - \beta + 1$, together with (18) and (25), gives

$$\begin{aligned}
&\sum_{k\leq\delta m^n} P(Z_n = k)P(S_k \geq k\epsilon_n)\\
&\leq C\sum_{k\leq\delta m^n} P(Z_n = k)\left(kP(X_1 \geq r^{-1}\epsilon_n k) + k^{(1-\beta)r}\epsilon_n^{-\beta r}\right)\\
&\leq C\left(L(\epsilon_n m^n)\epsilon_n^{-\beta}(m^n)^{1-\beta}\delta^{\gamma-\beta+1-\eta} + \delta^{(1-\beta)r+\gamma}(m^n)^{(1-\beta)r}\epsilon_n^{-\beta r}\right).
\end{aligned} \tag{37}$$

Choosing $r > 1$ and noting $\epsilon_n m^n b(m^n)^{-1} \to +\infty$, one can check that

$$(m^n)^{(1-\beta)r}\epsilon_n^{-\beta r}\frac{L(\epsilon_n m^n)}{\epsilon_n^{\beta}(m^n)^{(\beta-1)}} = o(1). \tag{38}$$

Then (35) follows readily if $\beta < 1$. The case of $\beta \geq 1$ can be proved similarly by applying (18) again with $r = \frac{\alpha\beta}{1-\alpha} + \beta + 1$ and $(1-t)r = 1$.

Similar reasonings also yields (36) by applying (3) with $\eta = \beta$. In fact, if $\beta < 1$, (18) and (25) imply

$$\begin{aligned}
&\sum_{k\geq A m^n} P(Z_n = k)P(S_k \geq k\epsilon_n)\\
&\leq C(1+o(1))L(\epsilon_n m^n)\epsilon_n^{-\beta}m^{n(1-\beta)}\int_A^\infty u\omega(u)du + CA^{(1-\beta)r}(m^n)^{(1-\beta)r}\epsilon_n^{-\beta r},
\end{aligned} \tag{39}$$

which, together with (38), proves (36) in the case of $\beta < 1$. Applying (3), (18) and (25) suitably also proves the case of $\beta \geq 1$. We omit the details here. $\square$

Lemma 3.2. *Assume that $\gamma > \beta - 1$, $\epsilon_n m^n b(m^n)^{-1} \to +\infty$ and $\epsilon_n \to +\infty$. Then there exits $\eta > 0$ small enough such that for any $0 < \delta < 1 < A$,*

$$\limsup_{n\to\infty} \left| m^{(\beta-1)n} \epsilon_n^{\beta} L(\epsilon_n m^n)^{-1} \sum_{k=\delta m^n}^{Am^n} P(S_k \geq \epsilon_n k) P(Z_n = k) - I_\beta \right| \leq C \left(\int_A^\infty u\omega(u)du + \delta^{\gamma-\beta+1-\eta} \right). \tag{40}$$

Proof. Using Theorem 9.3 in [7] for $\alpha < \beta$ and using Theorem 3.3 in [6] for $\alpha = \beta$, we have that

$$\lim_{n\to\infty} \sup_{x \geq x_n} \left| \frac{P(S_n \geq x)}{nP(X_1 \geq x)} - 1 \right| = 0.$$

holds for any x_n satisfying $nF(-x_n) = o(1)$ if $\alpha < \beta$ or $n(1 - F(x_n)) = o(1)$ if $\alpha = \beta$. Since $\epsilon_n m^n b(m^n)^{-1} \to \infty$, we have $m^n F(-\epsilon_n m^n) = o(1)$ if $\alpha < \beta$ and $m^n(1 - F(\epsilon_n m^n)) = o(1)$ if $\alpha = \beta$. In fact, if $\alpha < \beta$, we could denote by b^{-1} the inverse of b. Then $\epsilon_n m^n b(m^n)^{-1} \to \infty$ implies $\frac{m^n}{b^{-1}(\epsilon_n m^n)} \to 0$ and hence by (8) we have

$$m^n F(-\epsilon_n m^n) = \frac{m^n}{b^{-1}(\epsilon_n m^n)} b^{-1}(\epsilon_n m^n) F(-\epsilon_n m^n) \to 0.$$

If $\alpha = \beta$, the argument is similar. Define

$$\eta_n := \sup_{\delta m^n < k < Am^n} \sup_{x \geq \epsilon_n k} \left| \frac{P(S_k \geq x)}{kP(X_1 \geq x)} - 1 \right|.$$

Then one can check that $\eta_n = o(1)$ as $n \to \infty$. Thus as $n \to \infty$,

$$\begin{aligned} \sum_{k=\delta m^n}^{Am^n} P(Z_n = k) P(S_k \geq \epsilon_n k) &= (1 + o(1)) \sum_{k=\delta m^n}^{Am^n} kP(Z_n = k) P(X_1 \geq \epsilon_n k) \\ &= (1 + o(1)) \epsilon_n^{-\beta} \sum_{k=\delta m^n}^{Am^n} L(\epsilon_n k) k^{1-\beta} P(Z_n = k) \\ &= (1 + o(1)) \epsilon_n^{-\beta} \sum_{k=\delta m^n}^{Am^n} L(\epsilon_n k) k^{1-\beta} P(Z_n = k). \end{aligned} \tag{41}$$

Meanwhile, applying (3) with some $0 < \eta < \gamma - \beta + 1$ and (25) yields

$$L(\epsilon_n m^n)^{-1} m^{(\beta-1)n} \sum_{k<\delta m^n} L(\epsilon_n k) k^{1-\beta} P(Z_n = k) \leq C\delta^{\gamma-\beta+1-\eta} \tag{42}$$

and applying (3) with $\eta = \beta$ and (25) gives

$$L(\epsilon_n m^n)^{-1} m^{(\beta-1)n} \sum_{k>Am^n} L(\epsilon_n k) k^{1-\beta} P(Z_n = k) \leq (1 + o(1)) C \int_A^\infty u\omega(u)du. \tag{43}$$

Thus by Lemma 2.2, we have

$$\left| m^{(\beta-1)n} L(\epsilon_n m^n)^{-1} \sum_{k=\delta m^n}^{Am^n} L(\epsilon_n k) k^{1-\beta} P(Z_n = k) - I_\beta \right| \leq (1 + o(1)) C \left(\int_A^\infty u\omega(u)du + \delta^{\gamma-\beta+1-\eta} \right). \tag{44}$$

Then by (41), as $n \to \infty$,

$$
\begin{aligned}
&\left| m^{(\beta-1)n}\epsilon_n^{\beta} L(\epsilon_n m^n)^{-1} \sum_{k=\delta m^n}^{Am^n} P(S_k \ge \epsilon_n k) P(Z_n = k) - I_\beta \right| \\
&= \left| (1+o(1)) m^{(\beta-1)n} \sum_{k=\delta m^n}^{Am^n} L(\epsilon_n k) k^{1-\beta} P(Z_n = k) - I_\beta \right| \\
&\le (1+o(1))C \left(\int_A^\infty u\omega(u)du + \delta^{\gamma-\beta+1-\eta} \right).
\end{aligned}
$$

The desired result follows readily. $\square$

Proof of Theorem 1.5: *Letting $\delta \to 0$ and $A \to \infty$ in Lemmas (3.1) and (3.2) gives the theorem.* $\square$

3.2 Proof of Theorem 1.6

Recall that $l(x)$ is an asymptotic inverse of $x \mapsto J(x) = xb(x)^{-1}$ and $l(\epsilon_n^{-1}) = l_n$. If $\alpha < \beta$, we may write

$$
l(x) = x^{\frac{\alpha}{\alpha-1}} s'(x) \tag{45}
$$

for some slowly varying function s'. Note that Assumption B implies that

$$
\liminf_{x \to +\infty} s'(x) > 0. \tag{46}
$$

Lemma 3.3. *Assume that $1 < \alpha < 2$, $p_+ = 0$, $\gamma > \beta - 1$, $\epsilon_n m^n b(m^n)^{-1} \to +\infty$ and $\epsilon_n \to 0$. Then for any $0 < \delta < 1 < A$,*

$$
\sum_{1 \le k \le \delta l_n} P(Z_n = k) P(S_k \ge k\epsilon_n) \le C\delta^\gamma l_n^\gamma m^{-\gamma n}, \tag{47}
$$

$$
\sum_{k \le \delta m^n} P(Z_n = k) P(S_k \ge k\epsilon_n) \le C\delta^{\gamma+1-\beta-\eta} \epsilon_n^{-\beta} m^{(1-\beta)n} L(\epsilon_n m^n) + C l_n^\gamma m^{-\gamma n}, \tag{48}
$$

$$
\sum_{k \ge Am^n} P(Z_n = k) P(S_k \ge k\epsilon_n) \le C\epsilon_n^{-\beta} m^{(1-\beta)n} L(\epsilon_n m^n) + CA^{-2\gamma} l_n^\gamma m^{-\gamma n}, \tag{49}
$$

and for any A large enough,

$$
\begin{aligned}
&\sum_{Al_n < k \le Am^n} P(Z_n = k) P(S_k \ge k\epsilon_n) \\
&\le C(1 + A^{\gamma+1-\beta+\eta}) \epsilon_n^{-\beta} m^{(1-\beta)n} L(\epsilon_n m^n) + CA^{-2\gamma} l_n^\gamma m^{-\gamma n}.
\end{aligned} \tag{50}
$$

Proof. The proof will be divided into three parts.

Part 1: We shall prove (47) which can be obtained by noting (25) and

$$
\begin{aligned}
\sum_{1 \le k \le \delta l_n} P(Z_n = k) P(S_k > k\epsilon_n) &\le \sum_{1 \le k \le \delta l_n} P(Z_n = k) \\
&\le \frac{C}{m^{\gamma n}} \sum_{1 \le k \le \delta l_n} k^{\gamma-1}
\end{aligned}
$$

$$\le C\delta^\gamma l_n^\gamma m^{-\gamma n}.$$

Part 2: We shall first prove (48) and (50). Recall Corollary 1.6 of [15]: If $A_t^+ := E[X_1^t 1_{\{X_1\ge 0\}}] < \infty$ and $y^t \ge 4kA_t^+$ for some $1 \le t \le 2$, then for $x > y$

$$P(S_k \ge x) \le kP(X_1 > y) + (e^2 kA_t^+/xy^{t-1})^{x/2y}. \tag{51}$$

Thus if $s > 1$, $1 \le t < \beta$ and

$$k > \left(4E[X_1^t 1_{\{X_1\ge 0\}}]s^t\right)^{1/(t-1)} \epsilon_n^{t/(1-t)},$$

then

$$P(S_k \ge k\epsilon_n) \le kP(X_1 \ge s^{-1}k\epsilon_n) + C(\epsilon_n)^{-ts/2}k^{(1-t)s/2}. \tag{52}$$

Furthermore, (46) implies that there exists $A_l > 0$ such that (52) holds for $t = \alpha$ and all $k > A_l l_n$. Thus

$$\begin{aligned}
&\sum_{k\le \delta m^n} P(Z_n = k)P(S_k \ge k\epsilon_n) \\
&\quad \le \sum_{1\le k\le A_l l_n} P(Z_n = k)P(S_k \ge k\epsilon_n) + \sum_{A_l l_n < k \le \delta m^n} P(Z_n = k)P(S_k \ge k\epsilon_n) \\
&\quad =: I_1 + I_2.
\end{aligned} \tag{53}$$

Applying (25) again gives

$$\begin{aligned}
I_1 \le \sum_{1\le k\le A_l l_n} P(Z_n = k) &\le \frac{c}{m^{\gamma n}} \sum_{1\le k\le A_l l_n} k^{\gamma-1} \\
&\le CA_l l_n^\gamma m^{-\gamma n}.
\end{aligned} \tag{54}$$

Note that $l_n\epsilon_n \to \infty$. Applying (52) with $t = \alpha$, (3) with $\eta < \gamma - \beta + 1$ and (25), we have

$$\begin{aligned}
I_2 &\le \sum_{A_l l_n < k \le \delta m^n} P(Z_n = k)\left(kP(X_1 \ge s^{-1}k\epsilon_n) + C(\epsilon_n)^{-\alpha s/2}k^{(1-\alpha)s/2}\right) \\
&\le \frac{C}{m^{\gamma n}}\left(\sum_{A_l l_n < k \le \delta m^n} k^\gamma P(X_1 \ge s^{-1}k\epsilon_n) + \sum_{k > A_l l_n} (\epsilon_n)^{-\alpha s/2}k^{(1-\alpha)s/2+\gamma-1}\right) \\
&\le \frac{C}{m^{\gamma n}}\left(L(\epsilon_n m^n)\sum_{k\le \delta m^n} \epsilon_n^{-\beta}k^{\gamma-\beta}(k/m^n)^{-\eta} + \sum_{k > A_l l_n} (\epsilon_n)^{-\alpha s/2}k^{(1-\alpha)s/2+\gamma-1}\right) \\
&\le C\delta^{\gamma+1-\beta-\eta}\epsilon_n^{-\beta}m^{(1-\beta)n}L(\epsilon_n m^n) + CA_l^{-2\gamma}l_n^\gamma m^{-\gamma n}s'(\epsilon_n^{-1})^{-2\gamma},
\end{aligned} \tag{55}$$

where in the last inequality, we use (45), (46) and choose $s = \frac{4\gamma}{\alpha-1}$ which implies $(1-\alpha)s/2+\gamma = -\gamma$. Plugging (54) and (55) into (53), together with (46), gives (48). Replacing A_l and δ by A and modifying the last two steps in (55) accordingly, we immediately obtain (50).

Part 3: We shall prove (49). Note that $\epsilon_n m^n b(m^n)^{-1} \to +\infty$ implies $l_n \le m^n$. Using (52) with $s = \frac{4\gamma}{\alpha-1}$ and (3) with $\eta = \beta$, we have

$$\begin{aligned}
&\sum_{k\ge Am^n} P(Z_n = k)P(S_k \ge k\epsilon_n) \\
&\quad \le \sum_{k\ge Am^n} P(Z_n = k)\left(kP(X_1 \ge s^{-1}k\epsilon_n) + C(\epsilon_n)^{-\alpha s/2}k^{(1-\alpha)s/2}\right)
\end{aligned}$$

$$\leq C\left(\sum_{k\geq Am^n} P(Z_n=k)\epsilon_n^{-\beta}k^{1-\beta}L(s^{-1}k\epsilon_n)+\sum_{k>Al_n}(\epsilon_n)^{-\alpha s/2}k^{(1-\alpha)s/2+\gamma-1}m^{-\gamma n}\right)$$
$$\leq C\epsilon_n^{-\beta}m^{(1-\beta)n}L(\epsilon_n m^n)+CA^{-2\gamma}l_n^{\gamma}m^{-\gamma n}s'(\epsilon_n^{-1})^{-2\gamma},$$

where the second term in the last inequality is deduced according to similar reasonings for (55). Then (49) follows readily. □

Lemma 3.4. *Assume that* $1\leq\alpha<2$, $p_+>0$, $\gamma>\beta-1$, $\epsilon_n m^n b(m^n)^{-1}\to+\infty$ *and* $\epsilon_n\to\epsilon\in[0,\infty)$. *Then there exists* $\eta>0$ *small enough such that for any* $0<\delta<1$,

$$\sum_{k\leq\delta m^n} P(Z_n=k)P(S_k\geq k\epsilon_n)\leq C\delta^{\gamma-\beta+1-\eta}L(\epsilon_n m^n)\epsilon_n^{-\beta}m^{n(1-\beta)}. \tag{56}$$

Proof. The proof will be divided into three steps.

Step 1: Note that $p_+>0$ implies $\alpha=\beta$. We first prove that

$$P(S_k\geq\epsilon_n k)\leq C\left(kP(X_1\geq r^{-1}\epsilon_n k)+\epsilon_n^{-\beta}k^{(1-\beta)}L(\epsilon_n k)\right),\quad k\geq 1. \tag{57}$$

Recall (5). By Lemma in [19], we have for $k\geq1$ and $x>0$,

$$P(S_k\geq x)\leq Ck\left(P(|X_1|\geq x)+\frac{\mu(2;x)}{x^2}+\frac{|\mu(1;x)|}{x}\right). \tag{58}$$

(7) implies, for $1<\beta<2$,

$$\mu(2;x)=E[|X_1|^2\cdot 1_{\{|X_1|\leq x\}}]\leq cx^{2-\beta}L(x),\quad x>0. \tag{59}$$

On the other hand, according to (5.17), (5.21) and (5.22) in Chapter XVII in [9] as $x\to\infty$,

$$\frac{x}{\mu(2;x)}E[|X_1|\cdot 1_{\{|X_1|>x\}}]\to c\neq 0 \tag{60}$$

which, together with $E[X_1]=0$, yields for $1<\beta<2$,

$$|\mu(1;x)|=|E\left[X_1\cdot 1_{\{|X_1|\leq x\}}\right]|\leq E\left[|X_1|\cdot 1_{\{|X_1|>x\}}\right]\sim cx^{-\beta}L(x).$$

Thus for $1<\beta<2$,

$$|\mu(1;x)|\leq cx^{-\beta}L(x),\quad x>0.$$

Then according to (58), we obtain that (57) holds for $1<\beta<2$.

Step 2: We shall prove (57) for $\alpha=\beta=1$. By Theorem 1.2 in [15], we have

$$P(S_k\geq x)\leq kP(X_1>x)+P(x), \tag{61}$$

where

$$P(x)=\exp\left\{1-\left(1+\frac{k\mu(2;x)-kx\mu(x)}{x^2}\right)\cdot\log\left(\frac{x^2}{k\mu(2;x)}+1\right)\right\}.$$

By Assumption B,

$$P(x)=\frac{ek\mu(2;x)}{x^2+k\mu(2;x)}\leq\frac{ek\mu(2;x)}{x^2},$$

which, together with (61) and (59), gives that (57) holds.

Step 3: We shall prove (56). By using (57), (3) and (25) accordingly,

$$\begin{aligned}
&\sum_{k\le\delta m^n} P(Z_n=k)P(S_k\ge\epsilon_n k)\\
&\quad\le C\Big(kP(X_1\ge\epsilon_n k)+\epsilon_n^{-\beta}k^{1-\beta}L(\epsilon_n k)\Big)\\
&\quad\le C\epsilon_n^{-\beta}\sum_{k\le\delta m^n} P(Z_n=k)k^{1-\beta}L(\epsilon_n k)\\
&\quad\le CL(\epsilon_n m^n)\epsilon_n^{-\beta}\sum_{k\le\delta m^n} P(Z_n=k)k^{1-\beta}(k/m^n)^{-\eta}\\
&\quad\le CL(\epsilon_n m^n)\epsilon_n^{-\beta}m^{(-\gamma+\eta)n}\sum_{k\le\delta m^n} k^{\gamma-\beta-\eta}\\
&\quad\le C\delta^{\gamma-\beta+1-\eta}L(\epsilon_n m^n)\epsilon_n^{-\beta}m^{n(1-\beta)}.
\end{aligned}$$

We have completed the proof. □

Lemma 3.5. *Suppose that $1<\alpha<2$, $\gamma>\beta-1$, $\epsilon_n m^n b(m^n)^{-1}\to+\infty$ and $\epsilon_n\to\epsilon\in[0,\infty)$. If $\beta>\alpha$, we further assume that*

$$\lim_{n\to\infty}\chi_n=y\in[0,\infty). \tag{62}$$

Then there exists $\eta>0$ small enough such that for any $0<\delta<1$,

$$\limsup_{n\to\infty}\left|\frac{m^{(\beta-1)n}\epsilon_n^{\beta}}{L(\epsilon_n m^n)}\sum_{k>\delta m^n}P(S_k\ge\epsilon_n k)P(Z_n=k)-I_\beta\right|\le C\delta^{\gamma-\beta+1-\eta}. \tag{63}$$

Proof. First, if $1<\alpha<2$ and $p_+=0$, then by Theorem 9.2 in [7],

$$\lim_{k\to\infty}\sup_{x\ge x_k}\left|\frac{P(S_k\ge x)}{kP(X_1\ge x)}-1\right|=0 \tag{64}$$

holds for any $x_k=t(\frac{\beta-\alpha}{\alpha-1}\log k)^{\frac{\alpha-1}{\alpha}}b(k), t>0$. Define

$$\eta_n:=\sup_{k>\delta m^n}\sup_{x\ge\epsilon_n k}\left|\frac{P(S_k\ge x)}{kP(X_1\ge x)}-1\right|.$$

Then one can apply (64) with $x_k=k\epsilon_n$ to ensure $\eta_n=o(1)$. To apply (64) it suffices to show

$$\liminf_{n\to\infty}\frac{m^n\epsilon_n}{b(\delta m^n)(\ln(m^n))^{\frac{\alpha-1}{\alpha}}}\to+\infty. \tag{65}$$

In fact, since L and s are slowly varying functions, then for any $\eta,\eta'>0$, there exists $C_\eta, C_{\eta'}$ such that

$$L(l_n^{-1}b(l_n)m^n)\le C_\eta l_n^{-\eta}b(l_n)^\eta m^{\eta n}$$

and

$$\begin{aligned}
\frac{l_n^{\gamma-\beta}m^{(\beta-1-\gamma)n}b(l_n)^\beta}{L(l_n^{-1}b(l_n)m^n)} &\ge C_\eta\frac{l_n^{\gamma-\beta+\frac{\beta}{\alpha}+\eta-\frac{\eta}{\alpha}}}{m^{(\gamma-\beta+1+\eta)n}}\frac{s(l_n)^\beta}{s(l_n)^\eta}\\
&\ge C_\eta C_{\eta'}\frac{l_n^{\gamma-\beta+\frac{\beta}{\alpha}+\eta-\frac{\eta}{\alpha}-\beta\eta'-\eta\eta'}}{m^{(\gamma-\beta+1+\eta)n}}.
\end{aligned} \tag{66}$$

Since $\alpha < \beta$, then one could choose η, η' small enough such that

$$0 < \chi := \frac{\gamma - \beta + 1 + \eta}{\gamma - \beta + \frac{\beta}{\alpha} + \eta - \frac{\eta}{\alpha} - \beta\eta' - \eta\eta'} < 1. \tag{67}$$

Thus (62) and (66) imply

$$\limsup_{n\to\infty} \frac{m^{\chi n}}{l_n} \in (0, +\infty]. \tag{68}$$

We also note that for any $\eta'' > 0$,

$$\delta m^n \epsilon_n / b(\delta m^n) = \left(\frac{\delta m^n}{l^n}\right)^{\frac{\alpha-1}{\alpha}} \frac{s(l_n)}{s(\delta m^n)} \geq C_{\eta''} \left(\frac{\delta m^{\chi n}}{l_n}\right)^{\frac{\alpha-1}{\alpha}} (\delta m^{(1-\chi)n})^{\frac{\alpha-1}{\alpha}} \left(\frac{l_n}{\delta m^n}\right)^{\eta''}.$$

Choosing η'' small enough in above, together with (68) and (67), yields that (65) holds. We get that $\eta_n = o(1)$. The rest proof for the case of $1 < \alpha < 2$ and $\beta > \alpha$ is similar to Lemma 3.2. We omit it here.

When $1 < \alpha = \beta < 2$ and $p_+ = 1$, (64) holds for x_k satisfying $x_k/b(k) \to \infty$; see [20] and references therein. Obviously, in this case $\eta_n = o(1)$.

When $1 \leq \alpha = \beta < 2$ and $0 < p_+ < 1$, (64) holds for x_k satisfying $kP(X_1 > x_k) \to 0$ and $\frac{k}{x_k}\int_{-x_k}^{x_k} x dF(x) \to 0$; see Theorem 3.3 in [6]. By using (3), (8) and the fact $\epsilon_n m^n b(m^n)^{-1} \to \infty$, one can check that $\eta_n = o(1)$. Then the desired result can be proved similarly. □

Lemma 3.6. *Assume that* $1 < \alpha < 2$, $p_+ = 0$, $\gamma > \beta - 1$, $\epsilon_n m^n b(m^n)^{-1} \to +\infty$ *and* $\epsilon_n \to 0$. *Then*

$$\begin{aligned} V_I(\delta, A) &\leq \varliminf_{n\to\infty} m^{\gamma n} l_n^{-\gamma} \sum_{\delta l_n < k < A l_n} P(Z_n = k) P(S_k \geq k\epsilon_n) \\ &\leq \varlimsup_{n\to\infty} m^{\gamma n} l_n^{-\gamma} \sum_{\delta l_n < k < A l_n} P(Z_n = k) P(S_k \geq k\epsilon_n) \leq V_S(\delta, A), \end{aligned} \tag{69}$$

where

$$\begin{aligned} V_I(\delta, A) &= \varliminf_{u\to 0} u^{1-\gamma}\omega(u) \int_\delta^A u^{\gamma-1} P(U_s \geq u^{\frac{\alpha-1}{\alpha}}) du, \\ V_S(\delta, A) &= \varlimsup_{u\to 0} u^{1-\gamma}\omega(u) \int_\delta^A u^{\gamma-1} P(U_s \geq u^{\frac{\alpha-1}{\alpha}}) du. \end{aligned}$$

Proof. Define

$$H_2 = \{\delta l_n < k < A l_n : k = (\mathrm{mod}) d\}. \tag{70}$$

By Corollary 5 in [10] and (25), we have

$$\begin{aligned} &(1 + o(1)) d \inf_{u \leq A l_n m^{-n}} u^{1-\gamma}\omega(u) \sum_{k\in H_2} \frac{k^{\gamma-1}}{m^{\gamma n}} P(S_k \geq \epsilon_n k) \\ &\leq \sum_{k\in H_2} P(Z_n = k) P(S_k \geq \epsilon_n k) \\ &= (1 + o(1)) d \sum_{k\in H_2} m^{-n} \omega\left(\frac{k}{m^n}\right) P(S_k \geq \epsilon_n k) \\ &\leq (1 + o(1)) d \sup_{u \leq A l_n m^{-n}} u^{1-\gamma}\omega(u) \sum_{k\in H_2} \frac{k^{\gamma-1}}{m^{\gamma n}} P(S_k \geq \epsilon_n k). \end{aligned} \tag{71}$$

Recall (4). Then for any $\delta > 0$

$$\lim_{n\to\infty} \sup_{k\in H_2} |P(S_k \geq k\epsilon_n) - P(U_s \geq k\epsilon_n/b(k))| = 0.$$

Recall that $J(x) = xb(x)^{-1}$ and l is the asymptotic inverse function of J. Then as $n \to \infty$,

$$\begin{aligned}
\sum_{k\in H_2} k^{\gamma-1} P(S_k \geq k\epsilon_n) &= (1+o(1)) \sum_{k\in H_2} k^{\gamma-1} P\left(U_s \geq \frac{k\epsilon_n}{b(k)}\right) \\
&= (1+o(1)) l_n^\gamma \sum_{k\in H_2} \left(k l_n^{-1}\right)^{\gamma-1} P\left(U_\alpha \geq \frac{k\epsilon_n}{b(k)}\right) l_n^{-1} \\
&= (1+o(1)) d^{-1} l_n^\gamma \int_\delta^A u^{\gamma-1} P(U_s \geq u^{1-1/\alpha}) du, \qquad (72)
\end{aligned}$$

where the last equality follows from the facts that

$$\frac{k\epsilon_n}{b(k)} = \frac{k^{\frac{\alpha-1}{\alpha}}}{\epsilon_n^{-1} s(k)} \sim \frac{k^{\frac{\alpha-1}{\alpha}}}{J(l_n) s(k)} = \frac{s(l_n)}{s(k)} \left(\frac{k}{l_n}\right)^{1-1/\alpha} \quad \text{and} \quad \lim_{n\to\infty} \sup_{k\in H_2} \frac{s(l_n)}{s(k)} = 1.$$

Then letting $n \to \infty$ in (71) and (72) implies the desired result by noting the fact $l_n m^{-n} \to 0$. $\square$

Proof of (i) in Theorem 1.6: If $\chi_n \to 0$, then we have

$$l_n^\gamma m^{-\gamma n} m^{(\beta-1)n} \epsilon_n^\beta L(\epsilon_n m^n)^{-1} = o(1).$$

Thus combining (48) and Lemma 3.5 together and letting $\delta \to 0$ yield the desired result. $\square$

Proof of (ii) in Theorem 1.6: Recall H2 from (70). By taking A large enough in (49) and (50), we have

$$\begin{aligned}
\sum_{k\notin H_2} P(Z_n = k) P(S_k \geq k\epsilon_n) &= \left(\sum_{1\leq k\leq \delta l_n} + \sum_{Al_n < k < Am^n} + \sum_{k\geq Am^n} \right) P(Z_n = k) P(S_k \geq k\epsilon_n) \\
&\leq C(2 + A^{\gamma+1-\beta+\eta}) \epsilon_n^{-\beta} m^{(1-\beta)n} L(\epsilon_n m^n) \\
&\quad + C(A^{-2\gamma} + \delta^\gamma) l_n^\gamma m^{-\gamma n}. \qquad (73)
\end{aligned}$$

Since $\chi_n \to \infty$, we have $\epsilon_n^{-\beta} m^{(1-\beta)n} L(\epsilon_n m^n) = o(l_n^\gamma m^{-\gamma n})$. Thus

$$\overline{\lim_{n\to\infty}}\, l_n^{-\gamma} m^{\gamma n} \sum_{k\notin H_2} P(Z_n = k) P(S_k \geq k\epsilon_n) \leq C(A^{-2\gamma} + \delta^\gamma).$$

By (69), we further have

$$\begin{aligned}
V_I(\delta, A) &\leq \underline{\lim_{n\to\infty}}\, l_n^{-\gamma} m^{\gamma n} P(S_{Z_n} \geq Z_n \epsilon_n) \\
&\leq \overline{\lim_{n\to\infty}}\, l_n^{-\gamma} m^{\gamma n} P(S_{Z_n} \geq Z_n \epsilon_n) \leq C(A^{-2\gamma} + \delta^\gamma) + V_S(\delta, A). \qquad (74)
\end{aligned}$$

Letting $\delta \to 0$ and $A \to \infty$, together with the fact $V_I(\delta, A) \to V_I$ and $V_S(\delta, A) \to V_S$, yields (12). $\square$

Proof of (iii) in Theorem 1.6: Note that $\chi_n \to y \in (0, \infty)$ implies that

$$l_n^\gamma m^{-\gamma n} \sim y \epsilon_n^\beta m^{(\beta-1)n} L(\epsilon_n m^n)^{-1}.$$

Then the desired result follows from (56), (69), (50) and (63). □

Proof of (iv) in Theorem 1.6: Combining Lemmas 3.4 and 3.5 together and letting $\delta \to 0$ yield the desired result. We have completed the proof of Theorem 1.6. □

3.3 Proof of Theorem 1.16

First, note that $\int_0^\infty P\left(U_s \geq u^{\frac{\alpha-1}{\alpha}} x\right)\omega(u)du < \infty$. Then by (4), for any $\delta > 0$,

$$\lim_{n\to\infty} \sup_{k\geq \delta m^n} |P(S_k \geq \epsilon_n k) - P(U_s \geq \epsilon_n k/b(k))| = 0.$$

Thus

$$\sum_{k\geq\delta m^n} P(Z_n = k)P(S_k \geq \epsilon_n k) = (1+o(1)) \sum_{k\geq\delta m^n} P(Z_n = k)P(U_s \geq \epsilon_n k/b(k)).$$

Denote by $\bar{F}_s(x) = P(U_s \geq x)$. Then we have

$$\begin{aligned}
&\sum_{k\geq\delta m^n} P(Z_n = k)P(S_k \geq \epsilon_n k) \\
&= (1+o(1)) \sum_{k\geq\delta m^n} P(Z_n = k)\bar{F}_s\left(\epsilon_n m^n b(m^n)^{-1}\left(\frac{k}{m^n}\right)^{\frac{\alpha-1}{\alpha}} \frac{s(m^n)}{s(k)}\right) \\
&= (1+o(1))E\left[\bar{F}_s\left(\epsilon_n m^n b(m^n)^{-1}(W_n)^{\frac{\alpha-1}{\alpha}} \frac{s(m^n)}{s(W_n m^n)}\right)1_{\{W_n\geq\delta\}}\right] \\
&\to \int_\delta^\infty \bar{F}_s\left(u^{\frac{\alpha-1}{\alpha}} x\right)\omega(u)du.
\end{aligned}$$

On the other hand, by (22), as $n \to \infty$,

$$\sum_{k\leq\delta m^n} P(Z_n = k)P(S_k \geq \epsilon_n k) \leq \sum_{k\leq\delta m^n} P(Z_n = k) = (1+o(1))\int_0^\delta \omega(u)du. \tag{75}$$

Letting δ go to 0 yields the desired result.

3.4 Proofs of Theorem 1.13 and Corollary 1.14

We first prove Theorem 1.13. Applying (52) with $\epsilon_n = \epsilon$, $k > C_s\epsilon^{\frac{t}{1-t}} =: A(s,t,\epsilon)$ and $s = \frac{2\gamma+2}{t-1} > 1$ implies

$$\begin{aligned}
&m^{\gamma n}\sum_{k\geq 1} P(Z_n = k)P(S_n \geq \epsilon k) \\
&\leq C\sum_{k\geq 1} k^{\gamma-1}P(S_n \geq \epsilon k) \\
&\leq C\sum_{k\leq A(s,t,\epsilon)} k^{\gamma-1} + C\sum_{k>A(s,t,\epsilon)} k^{\gamma-1}P(S_n \geq \epsilon k) \\
&\leq CA(s,t,\epsilon)^\gamma + C\sum_{k>A(s,t,\epsilon)}\left(k^\gamma P(X_1 \geq s^{-1}\epsilon k) + \epsilon^{-ts/2}k^{(1-t)s/2+\gamma-1}\right)
\end{aligned}$$

$$\leq CA(s,t,\epsilon)^\gamma + C\sum_{k\geq 1} k^\gamma P(X_1 \geq s^{-1}\epsilon k) + \sum_{k>A(s,t,\epsilon)} \epsilon^{-ts/2}k^{-2}$$
$$< +\infty,$$

where the last equality follows from $\gamma < \beta - 1$ or the fact that $\sum_{k\geq 1} k^\gamma P(X_1 \geq s^{-1}\epsilon k)$ is finite because of $E[X_1^{1+\gamma} 1_{\{X_1>0\}}] < \infty$. Then by dominated convergence theorem, we have

$$m^{\gamma n}\sum_{k\geq 1} P(Z_n = k)P(S_k \geq \epsilon k) \to \sum_{k\geq 1} q_k P(S_k \geq \epsilon k), \tag{76}$$

which yields Theorem 1.13. To prove Corollary 1.14, note that by the same argument above, (76) also holds with $X_1 = m - Z_1$. Then Corollary 1.14 follows readily by applying (76) twice. □

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